Game theory &
Linear Programming

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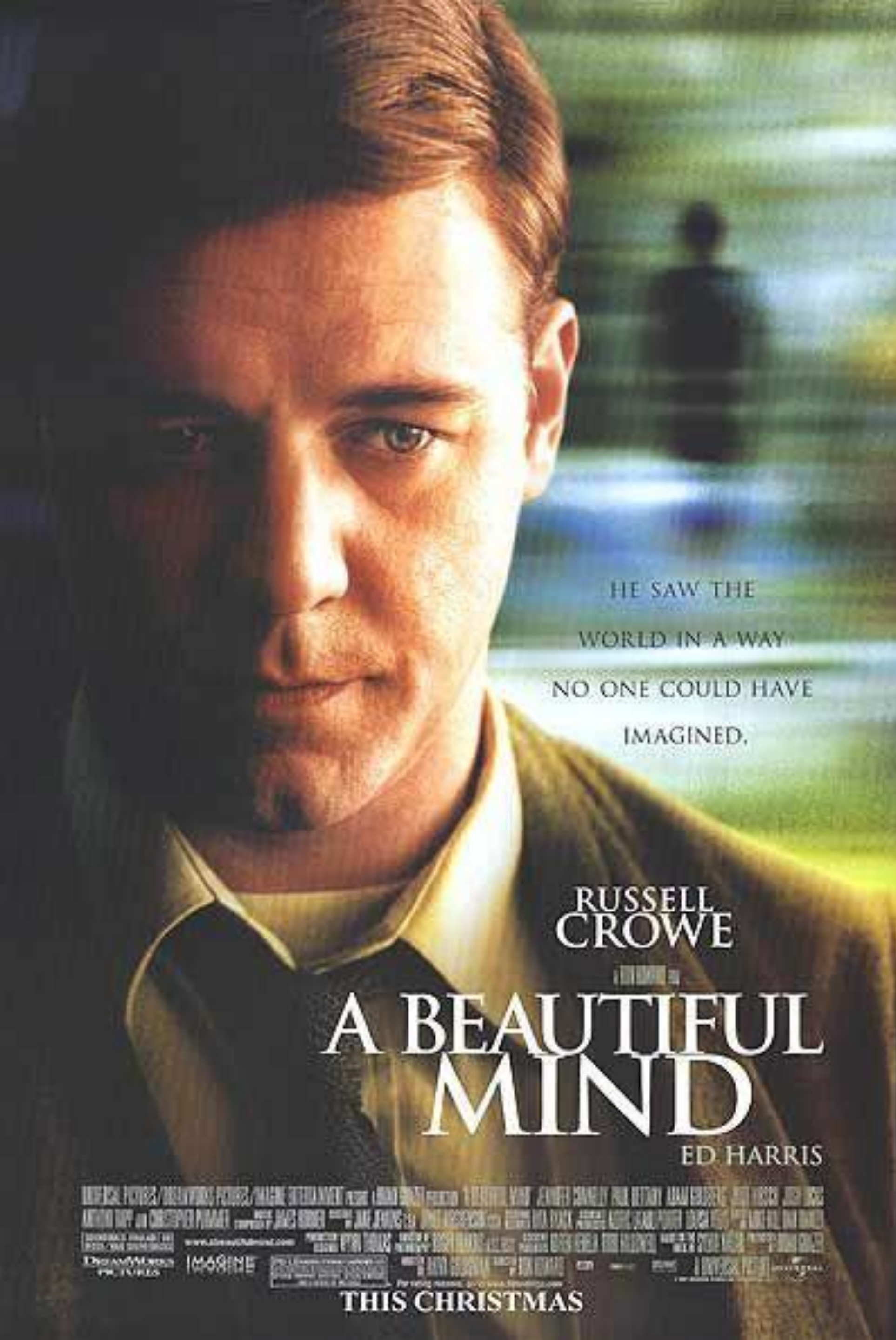
Outline

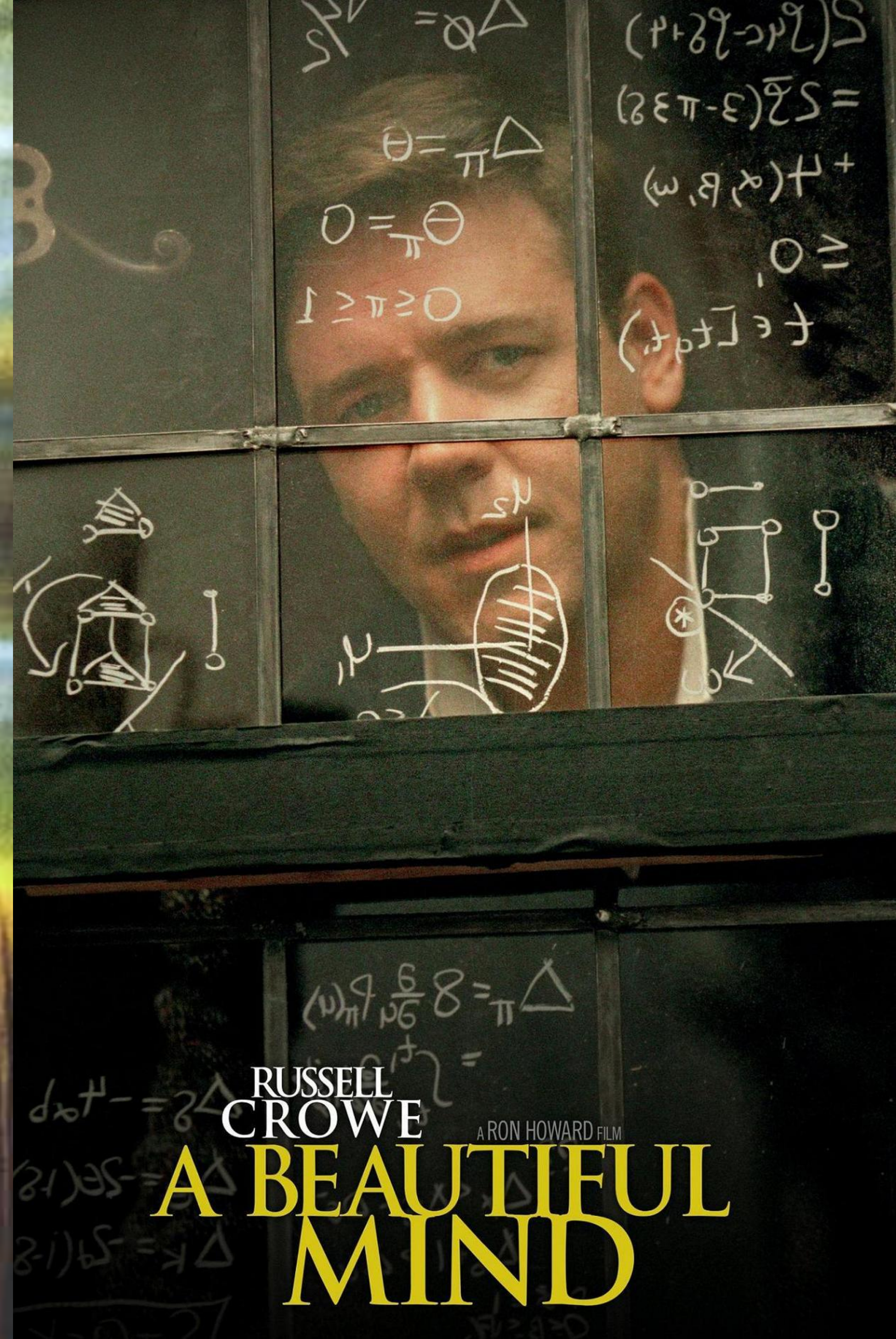
- Background
- Nash Equilibrium
- Zero-Sum Game
- How to Find NE using LP?
- Summary

Background: History of Game Theory

- John Von Neumann 1903 – 1957.
- Book - Theory of Games and Economic Behavior.

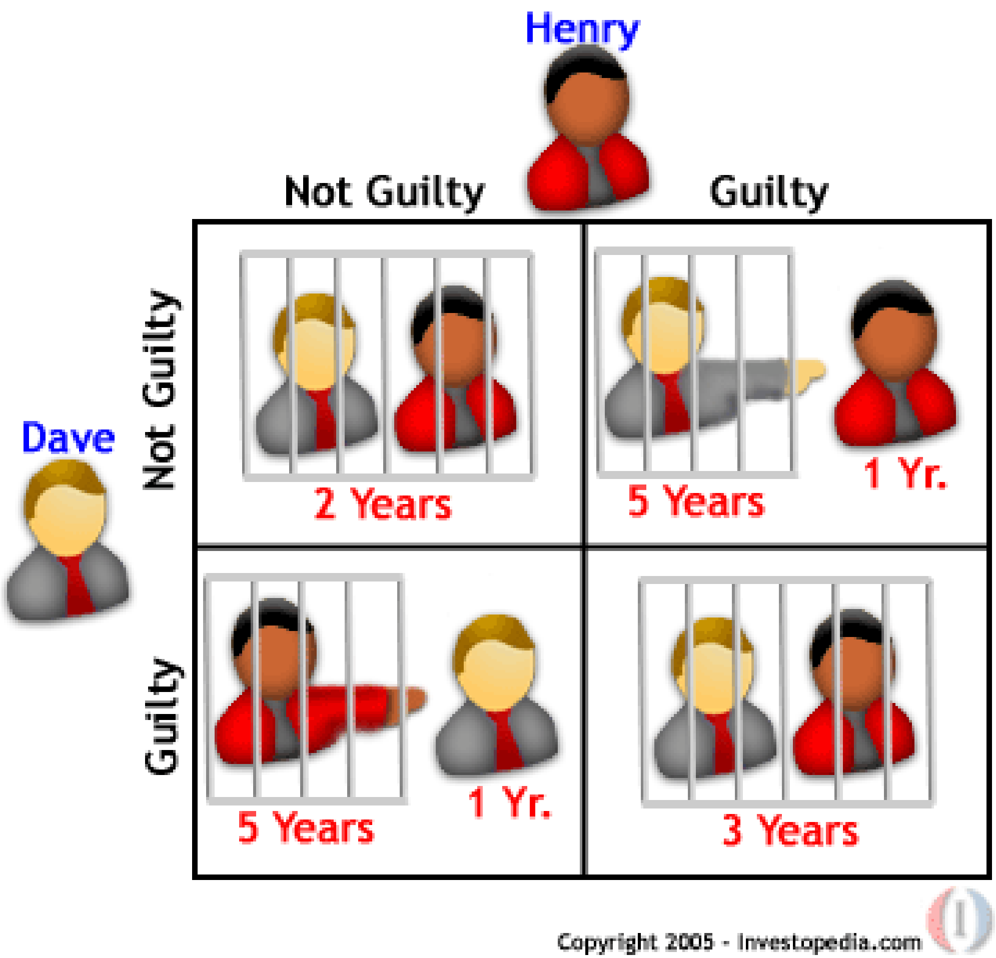
- John Forbes Nash 1928 –
- Popularized Game Theory with his Nash Equilibrium (Nobel prize).


HE SAW THE WORLD IN A WAY NO ONE COULD HAVE IMAGINED.
RUSSELL CROWE
A BEAUTIFUL MIND
ED HARRIS
THIS CHRISTMAS


RUSSELL CROWE
A RON HOWARD FILM
A BEAUTIFUL MIND

Background: Game theory

Background: Prison's Dilemma



Given you're Dave, what's the best choice?

Nash Equilibrium

Competition game

	Player 2 chooses '0'	Player 2 chooses '1'	Player 2 chooses '2'	Player 2 chooses '3'
Player 1 chooses '0'	*0, 0*	*2, -2*	*2, -2*	*2, -2*
Player 1 chooses '1'	*-2, 2*	*1, 1*	*3, -1*	*3, -1*
Player 1 chooses '2'	*-2, 2*	*-1, 3*	*2, 2*	*4, 0*
Player 1 chooses '3'	*-2, 2*	*-1, 3*	*0, 4*	*3, 3*

A competition game

A set of strategies is a Nash equilibrium if no player can do better by changing his or her strategy

Nash Equilibrium (Cont’)

- Nash showed (1950), that Nash equilibria (in mixed strategies) must exist for all finite games with any number of players.

- Before Nash's work, this had been proven for two-player zero-sum games (by John von Neumann and Oskar Morgenstern in 1947).

- Today, we’re going to find such Nash equilibria using Linear Programming for zero-sum game

Zero-Sum Game

- A *strictly competitive* or zero-sum game is a 2-player strategic game such that for each action $a \in A$, we have $u_1(a) + u_2(a) = 0$. *(u represents for utility)*
  - What is good for me, is bad for my opponent and *vice versa*

	A	B	C
1	30, -30	-10, 10	20, -20
2	10, -10	20, -20	-20, 20

Zero-Sum Game

- Mixed strategy
  - Making choice randomly obeying some kind of probability distribution
- Why mixed strategy? (→Nash Equilibrium)
- E.g. : P(1) = P(2) = 0.5; P(A) = P(B)=P(C)=1/3

	A	B	C
1	30, -30	-10, 10	20, -20
2	10, -10	20, -20	-20, 20

Solving Zero-Sum Games

- Let $A_1 = \{a_{11}, \ldots, a_{1n}\}$, $A_2 = \{a_{21}, \ldots, a_{2m}\}$
- Player 1 looks for a mixed strategy p
  - $\sum_i p(a_{1i}) = 1$
  - $p(a_{1i}) \geq 0$
  - $\sum_i p(a_{1i}) \cdot u_1(a_{1i}, a_{2j}) \geq r$ for all $j \in \{1, \ldots, m\}$
  - Maximize r!

- Similarly for player 2.

	A	B	C
1	30, -30	-10, 10	20, -20
2	10, -10	20, -20	-20, 20

Solve using Linear Programming

- What are the unknowns?
  - Strategy (or probability distribution): p
    - *$p(a_{11})$, $p(a_{12})$,…, $p(a_{1n-1})$, $p(a_{1n})$~n numbers*
    - *Denoted as $p_1, p_2, \ldots, p_{n-1}, p_n$*
  - Optimum Utility or Reward: r

$$x = \begin{pmatrix} r \\ p_1 \\ p_2 \\ \vdots \\ p_{n-1} \\ p_n \end{pmatrix}$$

- Stack all unknowns into a column vector
- Goal: maximize: $f^T x$ where $f = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \end{pmatrix}^T$

Solve Zero-Sum Game using LP

- What are the constraints?

$$x = \begin{pmatrix} r \\ p_1 \\ p_2 \\ \vdots \\ p_{n-1} \\ p_n \end{pmatrix}$$

$$\sum_{i=1}^{n} p_i = 1 \Rightarrow \begin{pmatrix} 0 & 1 & \cdots & 1 \end{pmatrix} x = 1$$

$$\sum_{i=1}^{n} p_i u_{ij} \geq r, \text{for all } j \in \{1, 2, \ldots, m\}, i.e.$$

$$r - u_{11} p_1 - u_{21} p_2 - \ldots - u_{n1} p_n \leq 0$$

$$r - u_{12} p_1 - u_{22} p_2 - \ldots - u_{n2} p_n \leq 0$$

$$\ldots$$

$$r - u_{1n} p_1 - u_{2n} p_2 - \ldots - u_{nb} p_n \leq 0$$

Solve Zero-Sum Game using LP

$$r - u_{11}p_1 - u_{21}p_2 - \ldots - u_{n1}p_n \leq 0$$

$$r - u_{12}p_1 - u_{22}p_2 - \ldots - u_{n2}p_n \leq 0$$

$$\ldots$$

$$r - u_{1n}p_1 - u_{2n}p_2 - \ldots - u_{nb}p_n \leq 0$$

$$\left(1; -U^T\right) x \leq 0 \Rightarrow Ax \leq 0$$

Let: $A_e=(0,1,\ldots,1)$

$A_{ie}=(1;-U^T)$

$f=(1,0,\ldots,0)^T$

The LP is:

maximize f^Tx

subject to:

$A_{ie}x<=0$

$A_ex=1$

Summary

- Zero-Sum Game → Mixed Strategy → NE
- Connections with Linear Programming

Thanks
Q&A